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Applications of the Wilson Flow in Lattice Gauge Theory

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Outline

1 Introduction
- Classical Conservation Laws
- Light Mesons
- Ward Identities and Anomaly
- Witten–Veneziano Formula
- Dilute Istanton Gas and Comparison with WV

2 Lattice
- Naive Discretization

3 Topological Charge
- Wilson Flow

4 Conclusions and Outlook

Classical Symmetries of the Action

- The Euclidean QCD action is:

$$S^{\mathrm{E}}_{\mathrm{QCD}} = \frac{1}{4} \int d^4x F_{\mu\nu} F_{\mu\nu} + \bar{\psi}(D + M)\psi \qquad (1)$$

  where $D = \gamma_\mu D_\mu$.

- For $N_f = 3$ and $M = 0$ the action is invariant under the group $U(3)_L \times U(3)_R$

$$\psi_L \to V_L \psi_L \qquad \bar{\psi}_L \to \bar{\psi}_L V_L^\dagger \qquad (2)$$

$$\psi_R \to V_R \psi_R \qquad \bar{\psi}_R \to \bar{\psi}_R V_R^\dagger \qquad (3)$$

Light Mesons

- The octect mesons can be interpreted as pseudo-Goldstone bosons
- u, d, s quark masses must be small

$$m_u, m_d \ll m_s \ll \Lambda_{\mathrm{QCD}}$$

- no parity partner in Nature
- Weinberg (1975):

$$m_{\eta'} < \sqrt{3} m_\pi$$

$\Rightarrow$ The $U(1)_A$ problem.

I	I_3	S	Meson	Quark content	Mass (MeV)
1	-1	0	π^-	$d\bar{u}$	140
1	1	0	π^+	$u\bar{d}$	140
1	0	0	π^0	$d\bar{d} - u\bar{u}/\sqrt{2}$	135
$\frac{1}{2}$	$\frac{1}{2}$	$+1$	K^+	$u\bar{s}$	494
$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	K^0	$d\bar{s}$	498
$\frac{1}{2}$	$-\frac{1}{2}$	-1	K^-	$s\bar{u}$	494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\bar{K}^0$	$s\bar{d}$	498
0	0	0	η	$\cos\theta\eta_8 + \sin\theta\eta_0$	547
0	0	0	η'	$\cos\theta\eta_8 - \sin\theta\eta_0$	958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\theta \simeq -11\,^\circ$$

Ward Identities

- Classical symmetry

$$S_F[\psi', \bar{\psi}'] = S_F[\psi, \bar{\psi}]$$

- QM symmetry

$$\int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{-S_F[\psi', \bar{\psi}']} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_F[\psi, \bar{\psi}]}$$

- Classical conservation laws must be translated in *Ward–Takahashi identities*:

$$\langle \delta S_F[\psi', \bar{\psi}'] \mathcal{O}[\psi', \bar{\psi}'] \rangle = \langle \delta \mathcal{O}[\psi', \bar{\psi}'] \rangle \quad (4)$$

- Transformations which leave the action invariant but not the partition function ⇒ *anomaly*

GMOR

$$\langle 2m \left(\partial_\mu j^{5a}_\mu - P^a \right) P^a \rangle = -\frac{\delta^{(4)}(x)}{N_f} \langle \bar{\psi}\psi \rangle \; (M = m \cdot \mathbb{1})$$

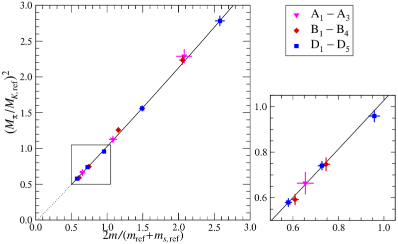
- *Chiral condensate*

$$\Sigma = \lim_{m \to 0} -\frac{\langle \bar{\psi}\psi \rangle}{N_f}$$

- *Gell-Mann–Oakes–Renner relation* (GMOR)

$$M_\pi^2 = \frac{2m\Sigma}{F^2} + \mathcal{O}(m^2) \quad (5)$$

[Del Debbio et al. 2006]



- Consequences of the GMOR:
  - In the chiral limit the pions become Goldstone bosons;
  - Quarks (u, d and s) are light with respect to the Λ_{QCD} scale;
  - At zero energy, Nambu-Goldstone bosons behave like free particles.

Witten–Veneziano

- Chiral abelian transformations are anomalous
- The anomaly is:

$$q(x) = -\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma} \operatorname{Tr}\{F_{\mu\nu}F_{\rho\sigma}\} \quad (6)$$

- The related Ward identity reads:

$$\langle \partial_\mu j^5_\mu(x) q(0)\rangle = 2m\langle P(x)q(0)\rangle + 2N_f\langle q(x)q(0)\rangle \quad (7)$$

- The topological charge is:

$$Q = \int d^4x q(x) \quad (8)$$

- The (Fourier transform) of the topological susceptibility is:

$$\chi(p) = \int d^4x e^{ipx}\langle q(x)q(0)\rangle \quad (9)$$

Witten–Veneziano

Large N_c expansion

- Using a technique called *large N_c expansion* $N_c \to \infty$ (keeping $g^2 N_c$ and N_f fixed), the η' mass can be written in series of:

$$u = N_f / N_c$$

- Witten and Veneziano in 1979 arrived at the following result, called *Witten–Veneziano formula*:

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2} \chi^{\mathrm{YM}}(0) + \mathcal{O}(u^2) \tag{10}$$

- Witten (1979): "We cannot ask whether the formula is correct, because it involves χ^{YM}, which we can neither measure nor calculate"
- In the chiral limit with $N_f / N_c \to 0$:
  - $U(1)_A$ is restored;
  - η' is a Nambu–Goldstone boson $\Rightarrow m_{\eta'} = 0$;
  - at first order in N_f / N_c, $m_{\eta'}^2 = \mathcal{O}(N_f / N_c)$ (note: $\chi^{\mathrm{YM}} = \mathcal{O}(1)$);

Dilute Istanton Gas and Comparison with WV

't Hooft (1976) proposed a solution of the $U(1)_A$ problem using *instantons*

't Hooft:

- Dilute Instanton Gas model;
- Introduced the topological charge of the system as $\nu = n - \bar{n}$;
- Prediction on the topological susceptibility:

$$\frac{\langle Q^2 \rangle}{VT} = 2e^{-S_0} K \cos\theta$$

Witten–Veneziano:

- "large N_c" expansion;

$$m^2_{\eta'} = \frac{2N_f}{F^2_\pi} \chi^{\text{YM}}(0) + \mathcal{O}(u^2)$$

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$$\frac{\langle Q^4 \rangle_{\text{con}}}{\langle Q^2 \rangle} = 1 \qquad \forall\theta$$

$$\frac{\langle Q^4 \rangle_{\text{con}}^{\text{YM}}}{\langle Q^2 \rangle^{\text{YM}}} \propto \frac{1}{N_c^2}$$

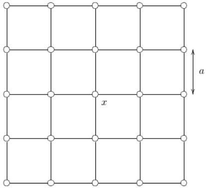
Lattice Regularization

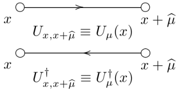
- We introduce of a 4-dim lattice, with the following requests on the lattice action, with lattice spacing a:
  1. reproduce the continuum action when $a \to 0$;
  2. invariant under the gauge symmetry $SU(3)$;
- The gauge field is introduced via a quantity defined between two lattice sites called link: $U_\mu(n)$;
- The field strenght tensor is defined with plaquettes:

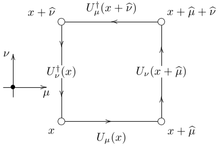
$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\nu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)$$

- The chiral symmetry is broken explicitly *(Wilson action)* to avoid the problem of *doublers*;
- The pure gauge action is given by:

$$S_G^W[U] = \frac{6}{g^2} \sum_P \left[1 - \frac{\text{Tr}}{6} (U_p + U_P^\dagger) \right] \qquad (11)$$







Neuberger Dirac Operator

- In 1982 Ginsparg and Wilson proposed a modified version of chiral symmetry on the lattice with the correct continuum limit;
- $U(1)_A$ anomaly recovered *à la Fujikawa* from the Jacobian of the new chiral transformation

$$a^4 q_N(x) = -\frac{\bar{a}}{2}\,\mathrm{Tr}\{\gamma_5 D_N(x,x)\} = \frac{1}{2}\,\mathrm{Tr}\{\hat{\gamma}_5\} \tag{12}$$

- Atiyah-Singer theorem on the lattice:

$$q_N(x) = \mathrm{index}(D_N) = n^- - n^+ \tag{13}$$

- for smooth gauge configurations the usual form of the anomaly is restored

Wilson Flow

- The Wilson flow generates smooth gauge configurations on the lattice.
- Differential equation with respect of *flow time*

$$V_t(x,\mu) = -g_0\left(\partial_{x,\mu} S_W(V_t)\right) V_t(x,\mu) \tag{14}$$

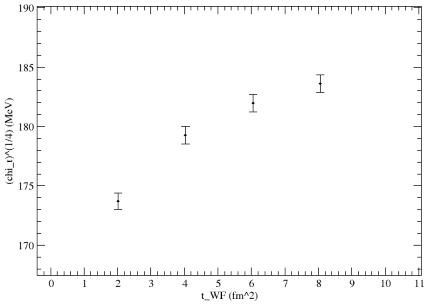
with the initial condition:

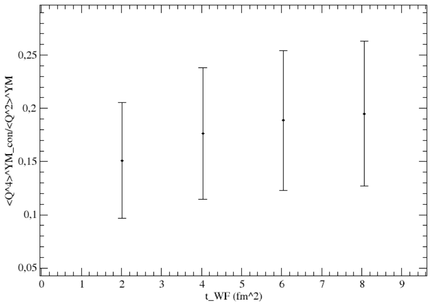
$$V_t(x,\mu)|_{t=0} = U(x,\mu) \tag{15}$$

- the topological charge is then defined as

$$q^t_{\mathrm{WF}}(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\{F^t_{\mu\nu}F^t_{\mu\nu}\} + \mathcal{O}(a^2) \tag{16}$$

- I have implemented a third order Runge–Kutta integrator on the group $SU(3)$ to integrate the WF equation



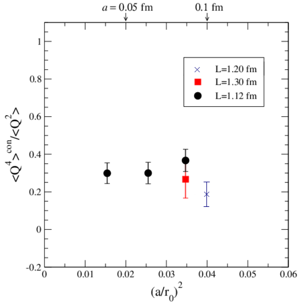


Results II

- At the greatest "Wilson flow time" measured:

$$(\chi^{\mathrm{YM}})^{-1/4} = 183.6 \pm 0.7\ \mathrm{MeV}$$
$$r = \frac{\langle Q^4 \rangle^{\mathrm{YM}}_{\mathrm{con}}}{\langle Q^2 \rangle^{\mathrm{YM}}} = (1.9 \pm 0.7) \cdot 10^{-1} \tag{17}$$

- the result is compatible with the values obtained with different methods.



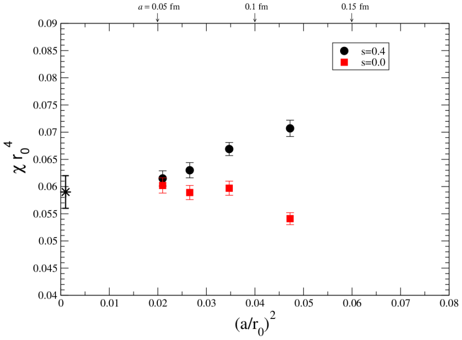
Conclusions

- A precise and unambiguous implementation of the Witten–Veneziano formula can be derived at the non-perturbative level in QCD
- I have implemented a RK integrator which allowed to compute χ^{YM} and $\langle Q^4 \rangle^{\mathrm{YM}}_{\mathrm{con}} / \langle Q^2 \rangle^{\mathrm{YM}}$
- Comparison with results obtained with different methods suggest that the sistematic errors are under control and the result has a significance of its own within is statistical error
- The (leading) QCD anomalous contribution to $m^2_{\eta'}$ explains the bulk of its large experimental value as conjectured by Witten and Veneziano in the setting of "large N_c" expansion.
- 't Hooft's dilute instanton gas model is inconsistent with this result

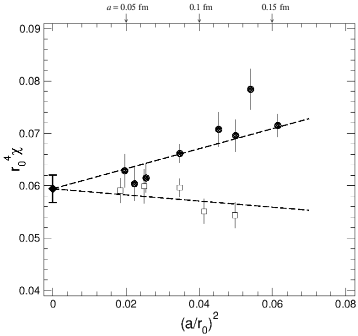
Backup

Discretization Errors I

[Del Debbio et al. (2005)]

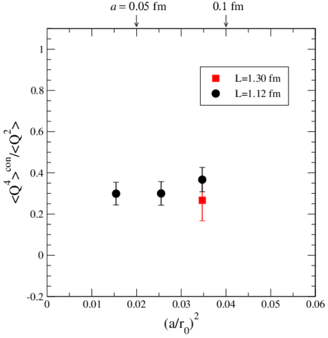


[Giusti et al. (2010)]

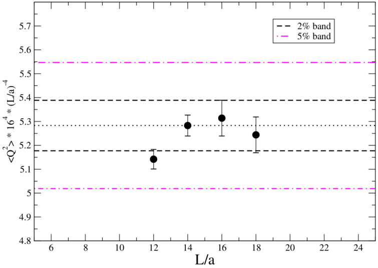


Discretization Errors II

[Giusti et al. (2010)]

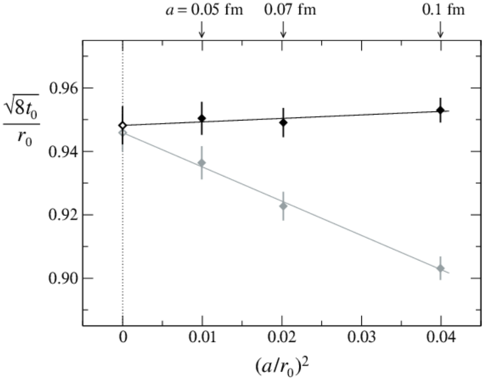


[Giusti et al. (2007)]



Discretization Errors III

[Lüscher (2010)]



Wilson Action and Ginsparg–Wilson Fermions

- Problem of *doublers* in naive implementation of fermions on the lattice
- Nielsen–Ninomyia theorem states that standard chiral $\{\gamma_5, D\} = 0$ symmetry can not be implemented on the lattice without givin up some important property (e.g. locality)
- Wilson proposed a modification of S_{QCD} which is free of doublers but breaks explicitly chiral symmetry
- In 1982 Ginsparg and Wilson propose to break chiral symmetry

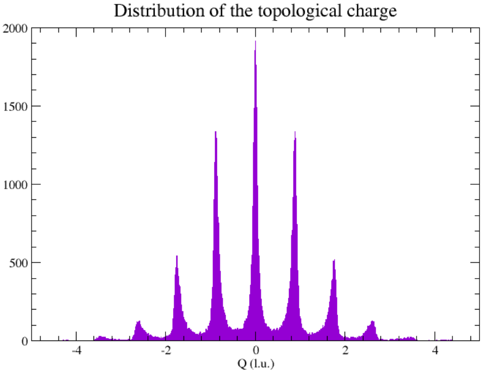
$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

- It is possible to define an exact symmetry at finite cutoff:

$$\begin{aligned} \psi &\to \psi'(x) = e^{i\beta\hat{\gamma}_5}\psi(x) \\ \bar{\psi} &\to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\beta\gamma_5} \end{aligned} \tag{18}$$

where $\hat{\gamma}_5 = \gamma_5(1 - aD)$

Distribution of the topological charge



Integration Step

- The Wilson flow equation can be written as:

$$\dot{V}_t = Z(V_t)V_t \tag{19}$$

- The integration from time t to $t + \varepsilon$ is given by:

$$\begin{cases} W_0 = V_t \\ W_1 = \exp\left\{\frac{1}{4}Z_0\right\} W_0 \\ W_2 = \exp\left\{\frac{8}{9}Z_1 - \frac{17}{36}Z_0\right\} W_1 \\ V_{t+\varepsilon} = \exp\left\{\frac{3}{4}Z_2 - \frac{8}{9}Z_1 + \frac{17}{36}Z_0\right\} W_2 \end{cases} \tag{20}$$

where

$$Z_i = \varepsilon Z(W_i) \qquad (\text{for } i = 0, 1, 2) \tag{21}$$